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The wellordering on positive braids

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Abstract

This paper studies Artin's braid monoids using combinatorial methods. More precisely, we investigate the linear ordering defined by Dehornoy. Laver has proved that the restriction of this ordering to positive braids is a wellordering. In order to study this order, we develop a natural wellordering $\ll$ on the free monoid on infinitely many generators by representing words as trees. Our construction leads to a (new) normal form for (positive) braids. Our main result is that the restriction of our order $\ll$ to the normal braid words coincides with the restriction of Dehornoy's ordering to positive braids. Our method gives an alternative proof of Laver's result using purely combinatorial arguments and gives the order type, namely $\omega^{\omega^{\omega}}$. © 1997 Elsevier Science B.V.

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0. Introduction

The n strand braid group, traditionally denoted as B_n, is introduced as the abstract group with $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ subject to the relations

$$\begin{cases} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |j-i| \geq 2. \end{cases} \tag{1}$$

We know (see for instance [1]) that B_n is the group of isotopy classes of n strand braids in the intuitive sense, the product being the concatenation of the strands, the generator σ_i corresponding to the elementary crossing of the strands i and $i+1$.

Dehornoy has constructed in [3] a *linear order* on the braids. This order $<$ is characterized by the fact that a braid β verifies $\beta > 1$ if and only if there exists a decomposition of β of the form

$$\beta = \beta_0\sigma_i\beta_1\sigma_i \ldots \sigma_i\beta_k,$$

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where $\beta_0, \beta_1, \dots, \beta_k$ admit decompositions where only $\sigma_{i+1}, \sigma_{i+2}, \dots$ (and their inverses) appear.

Using arguments of distributive algebra, Laver has shown in [7] that the restriction of the order $<$ to the submonoid B_n^+ is a wellordering. This is an application of his result that the order $<$ extends the partial order defined in [4] using Higman's theorem of [6].

In this paper we shall describe the properties of the order $<$, give a direct characterization of the wellordering $< \upharpoonright B_n^+$ and compute the associated ordinal, which is $\omega^{\omega^{n-2}}$.

Let Σ_n^* be the free monoid on the alphabet of letters $\{1, 2, \dots, n-1\}$. Under the coding ϕ that maps every letter i to the corresponding generator σ_i, every positive braid is an equivalence class on Σ_n^* relative to the congruence $\equiv$ generated by the braid relations.

Definition (*Special relation*). A binary relation $\lhd$ on Σ_n^* is *special* if it is compatible with the congruence $\equiv$, antisymmetric, transitive, compatible with left translations (i.e., $A \lhd B$ implies $C.A \lhd C.B$) and, moreover, satisfies the following condition (♣):

$$A.B \lhd a \dots b \text{ holds for } a \leq b, A \text{ in } \{a+1, a+2, \dots, b\}^*, B \text{ in } \{1, 2, \dots, b-1\}^*$$

It is clear that any special relation $\lhd$ on Σ_n^* induces a strict order on B_n^+ that extends the prefix ordering ($A \lhd A.B$ always holds), that $\varepsilon \lhd 1 \lhd 2 \lhd 3 \cdots \lhd (n-1)$ holds and that $a^i \lhd a^j$ holds if and only if $i < j$ does.

Using the terminology of the special relations, the above-mentioned results can be stated as

Proposition 1 (Dehornoy [3]). *For every integer n, there exists a special relation on Σ_n^*.*

Proposition 2 (Laver [7]). *For every integer n, if $\lhd$ is the special relation on Σ_n^* whose existence is asserted above, then $\lhd$ extends the subword ordering, and therefore $\lhd$ induces a wellordering on B_n^+ (by Higman's theorem of* [6]).

These results are established using properties of distributive algebra (the study of the operations that satisfy the left self-distributivity identity). Our main result here is the following refinement of Proposition 2, that we shall prove using purely combinatorial methods.

Proposition 3. *For every integer n, there exists at most one special relation on Σ_n^*. Such a relation extends the subword ordering, and it induces a wellordering of type $\omega^{\omega^{n-2}}$ on B_n^+.*

The main idea of this work is to associate a *tree structure* with every word, and to define a linear ordering $\ll$ of words in terms of the associated trees (actually, a lexicographic ordering of the trees). The key result is that the braid α precedes the braid β with respect to the order $<$ if and only if $u \ll v$ holds, where u is the $\ll$-minimal representative of α and v is the $\ll$-minimal representative of β. So, if the words that are $\ll$-minimal in their class are called *normal* (there is exactly one normal word for every braid), then $\alpha < \beta$ holds if and only if the normal decomposition of α precedes the normal decomposition of β in the order $\ll$: there is an isomorphism between $<$ and the restriction of $\ll$ to normal words. Since the relation $\ll$ on normal words of Σ_n^* is a wellordering of type $\omega^{\omega^{n-2}}$, it is easy to conclude. The main result above, that we call *coincidence* (between $<$ and $\ll$), is proved by induction along the wellordering $\ll$.

1. A wellordering on the free monoids

The empty word of Σ_n^* is denoted by ε. Single letters of the alphabet are denoted by a, b, c, d. And arbitrary words (including the empty word and letters) are denoted by A, B, C, D. For any two letters a, b, let us denote by $\mathscr{W}_{a,b}$ the set of words formed with the letters between a and b that end with the letter b. For instance, the word 5.3.4.3 belongs to $\mathscr{W}_{a,3}$ for any $a \geq 5$.

We are going to construct an isomorphism between Σ_n^* and a family $\mathscr{T}_n$ of trees of height n. A tree of $\mathscr{T}_n$ is presented as a finite set of lists of integers, its *addresses*. An address describes the path from the root to a node of the tree.

Definition (*Tree*). An *address* is a finite sequence of positive integers. The empty address is denoted by Λ. The addresses are written typically as u, v, w, x, y, z. A *tree* A of *height* n is a set of addresses of the form

$$\{\Lambda\} \cup \{kw; w \in A_k\} \cup \cdots \cup \{2w; w \in A_2\} \cup \{1w; w \in A_1\},$$

where $n > 1$, $k \geq 1$ and $A_k, \ldots, A_2, A_1$ trees of height $n-1$. The unique tree of height 1 is $\{\Lambda\}$. For every address w in A, the *degree of* w *in* A is the largest integer i such that the address wi is in A. The set of all the trees of height n is written $\mathscr{T}_n$. The addresses of length $n-1$ in a tree A of $\mathscr{T}_n$ are the *leaves* of A. Trees are written typically A, B, C, D and they admit a geometrical representation associated with the prefix ordering of the addresses.

Example. The set of addresses

$$\{\Lambda, 3, 31, 311, 2, 22, 221, 21, 211, 1, 13, 131, 12, 122, 121, 11, 112, 111\}$$

is a tree of $\mathscr{T}_4$. This tree is represented by Fig. 1.

The root has degree 3. The eight leaves of this tree are

$$311, 221, 211, 131, 122, 121, 112, 111.$$

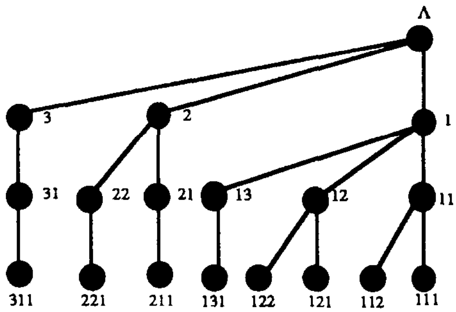

Fig. 1.

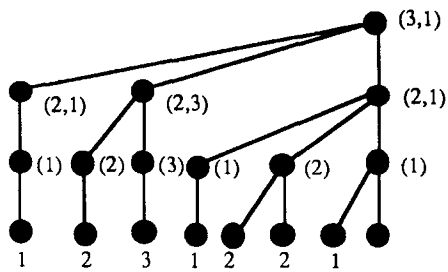

Fig. 2.

We associate with every tree a unique word. The construction of this word uses a colouring of each node of the tree by a list of letters.

Definition (*Domain and word*). Let A be a tree of $\mathcal{T}_n$. Let w be an address in A. The *domain of* w *in* A is a sequence of consecutive letters. The domain of Λ is $(n-1,\dots,2,1)$ and if the domain of an address x is $(c_i,\dots,c_1)$ then the domain of the address xk is $(c_{i-1},\dots,c_2,c_1)$ if k is odd and $(c_2,c_3,\dots,c_i)$ if k is even. Let xk be a leaf in A. The *word of* xk *in* A is the unique letter of the domain of the address x except for the rightmost leaf 1^{n-1} of which the word is empty. The *word of* w *in* A is the word formed by the concatenation of the words of all the leaves under the address w. The *word of the tree* A is the word of its root.

In the figures of trees, we shall use the notation (a,b) to stand for the domain with sequence of consecutive letters from a to b.

Example. The address 1 has domain $(2,1)$ and its word is 1.2.2.1 in Fig. 2. The word of this tree is 1.2.3.1.2.2.1.

We have constructed for every tree a word associated with it. Conversely, for every word A there is one and only one tree such that the word of this tree is A.

Lemma 4 (Representation). *The correspondence between the words in* Σ_n^* *and the trees in* $\mathcal{T}_n$ *is a bijection.*

Proof. Induction on the length of the word A. The tree associated with the empty word of Σ_n^* is a single branch of height n. Considering the unique tree associated with a word A, for any letter c there is one and only one way to add a branch on the left of this tree such that the word of the leaf of this new branch is c. This branch must be added under the lowest inner node on the left of the tree whose domain contains the letter c. $\square$

To construct the tree of a given word, we start with a single branch and successively add on the left the letters of the word, with the above rule that a new letter is connected to the lowest possible node that *accepts* it in the sense that its domain contains the letter.

Example. The tree of $\mathscr{T}_5$ associated with the word 1.3.2.2.1.2 of Σ_5^* is constructed in Fig. 3.

Definition (*Before*). Denote by $\sqsubset$ the lexicographic order on the addresses. The relations $123 \sqsubset 211$ and $121 \sqsubset 122$ hold. The tree A is *before* the tree B if the list of the leaves of A is less than the list of leaves of B in the lexicographic order according to $\sqsubset$. The word A is *before* the word B, written $A \ll B$, if the tree A is before the tree B.

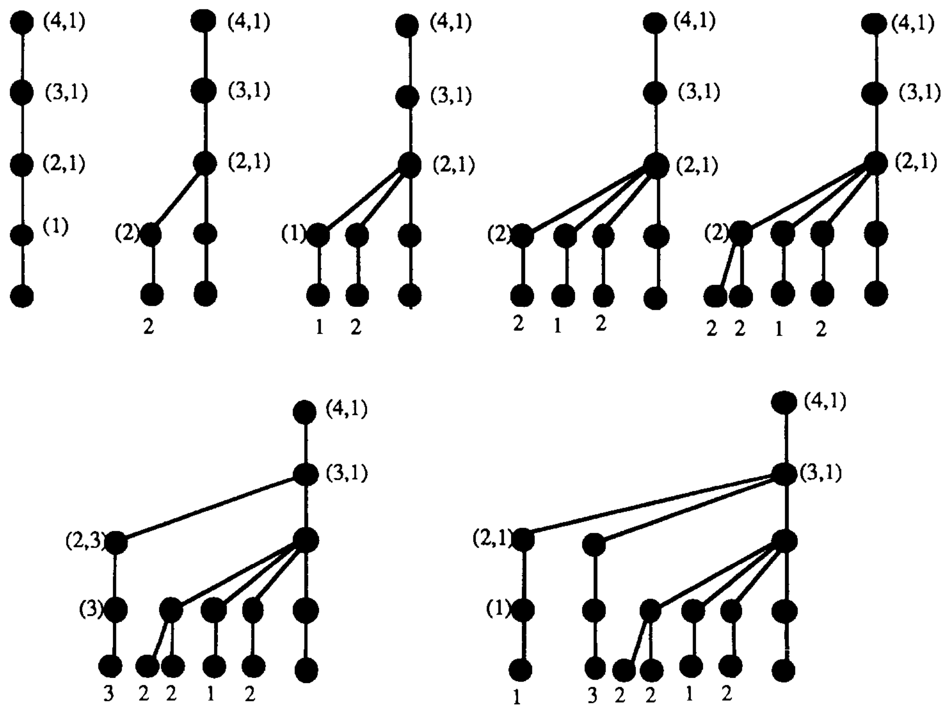

Fig. 3.

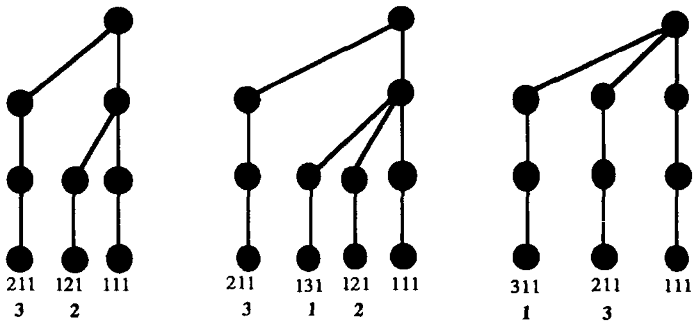

Fig. 4.

Geometrically, a tree is before another if and only if the first one is "thinner" than the second one at the root or their subtrees are ordered lexicographically from left to right.

Example. The word 3.2 is before the word 3.1.2, which is before 1.3. (see Fig. 4).

Proposition 5 (Wellordering). *Let n be an integer greater than 2.*

(i) *The order $\ll$ on Σ_n^* is a wellordering of type $\omega^{\omega^{n-2}}$.*

(ii) *The immediate successor of a word A in the order $\ll$ is the word $A.1$.*

(iii) *The order $\ll$ is compatible with left translations.*

(iv) *The order $\ll$ extends the subword order.*

Proof. Points (i) and (ii) are obvious.

(iii) It is sufficient to show that, for any words A,B and any letter c, the relation

$A \ll B \Leftrightarrow c.A \ll c.B$ holds.

Let $(x_l,\dots,x_1)$ be the list of leaves of A. Let $(y_m,\dots,y_1)$ be the list of leaves of B. Let x_{l+1} be the new leaf of $c.A$ and y_{m+1} be the new leaf of $c.B$.

Suppose $x_l = y_m$. By definition the nodes on the the left part of A and B are equal two by two and by construction $x_{l+1} = y_{m+1}$ holds.

Suppose $x_l \sqsubset y_m$. The address x_l begins with wi and y_m begins with wj where $i<j$. If c is not in the domain of w, then the addresses x_{l+1} and y_{m+1} are equal by construction. Else, if c is in the domain of w but not in the domain of wi, then the address x_{l+1} is of the form $w(i+1)1^k$ and $x_{l+1} \sqsubset y_m \sqsubset y_{m+1}$ holds. Else, if c is in the domain of wi, then the address x_{l+1} begins with wi and $x_{l+1} \sqsubset y_m \sqsubset y_{m+1}$ hold. In all cases, the $\sqsubset$-lexicographical order is preserved. The converse is obvious since $\ll$ is a linear order.

(iv) First, it is obvious by definition that $\ll$ extends the suffix order, i.e., $A \ll c.A$. Then, by induction on the integer k, the relation $a_k \dots a_1 \ll C_k.a_k \dots C_1.a_1.C_0$ holds for all words $C_k,\dots,C_0$ since $\ll$ is transitive and is compatible with left translations. □

Observe that the order $\ll$ is not compatible with right translations as $1 \ll 2$ and $1.2 \gg 2.2$ hold (see Fig. 5).

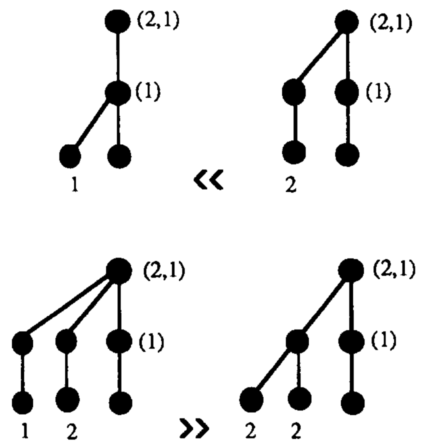

Fig. 5.

2. Application to positive braids

Any positive braid in B_n can be viewed as an equivalence class of words of Σ_n^*. In this way, one can define a normal form for any positive braid.

Definition (*Normal form*). For any word A, let us denote by $\|A\|$ the $\ll$-minimal element of the class of A. Say that A is *normal* if A is $\|A\|$.

Then, the order $\ll$ restricted to normal form words induces a wellordering on positive braids. In the sequel, we will see that this induced wellordering preserves all properties of the Proposition 5 (wellordering).

Definition (*Word Π*). Let A be a word. Write A^+ for the image of A by the morphism mapping every letter a to the corresponding letter $(a+1)$ and by A^- the image (if it exists) under the inverse morphism. Let a, b be two letters with $a \leq b$. The word $\Pi_{a,b}$ is the word of $\mathscr{W}_{a,b}$

$$a.(a+1)\ldots b.$$

The word $\Pi_{b,a}$ is the word of $\mathscr{W}_{b,a}$

$$b.(b-1)\ldots a.$$

The relation $a.(a+1).a \equiv (a+1).a.(a+1)$ then implies the relations

$$\begin{aligned} \Pi_{a,a+1}.a &\equiv a^+.\Pi_{a,a+1}, \\ \Pi_{a+1,a}.(a+1) &\equiv (a+1)^-.\Pi_{a+1,a}, \end{aligned}$$

a property that is generalized in the

Lemma 6 (Shifting). *Let a, b be two letters and C be a word.*

(i) *For $a < b$ and C in $\{a, \ldots, b-1\}^*$, $\Pi_{a,b}.C \equiv C^+.\Pi_{a,b}$ holds.*

(ii) *For $a > b$ and C in $\{a, \ldots, b+1\}^*$, $\Pi_{a,b}.C \equiv C^-.\Pi_{a,b}$ holds.*

Proof. It is sufficient to verify the property when C is a letter c.

(i) For $a \leq c < b$, we obtain

$$\begin{aligned}
\Pi_{a,b}.c &= \Pi_{a,c-1}.c.(c+1).\Pi_{c+2,b}.c \\
&\equiv \Pi_{a,c-1}.c.(c+1).c.\Pi_{c+2,b} \\
&\equiv \Pi_{a,c-1}.(c+1).c.(c+1).\Pi_{c+2,b} \\
&\equiv (c+1).\Pi_{a,c-1}.c.(c+1).\Pi_{c+2,b} \\
&\equiv c^{+}.\Pi_{a,b}.
\end{aligned}$$

(ii) For $a \geq c > b$, the computation is symmetrical. □

We are now going to characterize a set of words called *reducible*. Such a word is never normal because one can always construct another word that is equivalent and strictly less (w. r. to $\ll$). In order to introduce this notion, we need to define a convenient decomposition of words. This decomposition depends on the geometry of left part of the associated trees.

Definition (*Decomposition*). Denote by τ the partial mapping that maps any address of the form $x(i+2)1^j$ to the address $x(i+1)$. Let $a.A$ be a word with leftmost leaf w. The *decomposition* of the word $a.A$ is, if it makes sense, the sequence of words (a, B, C, D), where B is the word of the address $\tau(w)$, C is the word of the address $\tau^2(w)$ and D is the word that gives $A = B.C.D$.

Since the morphism τ is not defined on "right branch addresses" of the form 1^j, some words admit no decomposition. At this point, one can separate the words in two types.

Definition (*Type*). A word A has *type* 1 if one of the following holds:

(i) A has the form 1^i with $0 \leq i$,

(ii) A begins with at least twice the same letter (A has the form $a^2.B$),

(iii) A has the form $\Pi_{a,b}.B$ with B in the submonoid $\{1, \dots, b-1\}^*$.

The word A has *type* 2 otherwise.

By complementation, one obtains:

Lemma 7 (Type 2 decomposition). *Let $a.A$ be a word of type 2. Then the decomposition (a, B, C, D) of $a.A$ exists and one of the following holds*:

(i) *The word B is in $\mathscr{W}_{a+1,b}$ with $b > a$ and C is in $\mathscr{W}_{b-1,a-j}$ with $j \geq 0$.*

(ii) *The word B is in $\mathscr{W}_{a-1,b}$ with $a > b$ and C is in $\mathscr{W}_{b+1,a+j}$ with $j \geq 0$.*

Definition (*Reducibility*). A word $A'.a.A$ is *reducible* if $a.A$ has type 2 and for its decomposition (a, B, C, D) one of the following holds:

(i) The word B is in $\mathscr{W}_{a+1,b}$ with $b > a$ and B does not contain the letter $(a+1)$.

(ii) The word B has the form $\Pi_{a+1,b}$ with $b > a$, and the word C has the form $C_1.C_2$, where C_1 is in $\mathscr{W}_{a,b-1}$ and C_2, does not contain the letter $(b-1)$.

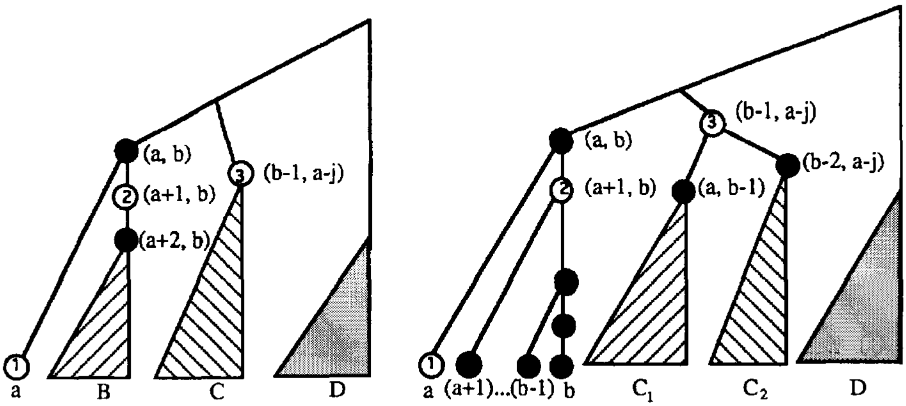

Fig. 6.

(iii) The word B is in $\mathcal{W}_{a-1,b}$ with $a > b$, and B does not contain the letter $(a-1)$.

(iv) The word B has the form $\Pi_{a-1,b}$ with $a > b$, and the word C has the form $C_1.C_2$ where C_1 is in $\mathcal{W}_{a,b+1}$ and C_2 does not contain the letter $(b+1)$ (Fig. 6).

The geometrical characterization of reducible words enables us to prove that those words cannot be $\ll$-minimal in their respective classes.

Lemma 8 (Reduction). *For any reducible word A, there exists a word $\Gamma(A)$ in the class of A that satisfies $\Gamma(A) \ll A$. So, reducible words cannot be normal.*

Proof. Let $A.a.B.C.D$ be a reducible word where $a.B.C.D$ is the smallest reducible suffix. Let us consider the cases of reducibility.

(i) The word B is in $\mathcal{W}_{a+1,b}$ with $b > a$ and does not contain the letter $(a+1)$. Let $\Gamma(A.a.B.C.D)$ be the word

$$A.B.a.C.D.$$

For the equivalence, as B does not contain the letter $(a+1)$, it commutes with the letter a. For the order, the left branch of letter a is inserted on the left of the tree of C and as C is in $\mathcal{W}_{b-1,a-j}$ with $j \geq 0$, the tree of C accepts this letter a (Fig. 7).

(ii) The word B has the form $\Pi_{a+1,b}$ with $b > a$, and the word C has the form $C_1.C_2$, where C_1 is in $\mathcal{W}_{a,b-1}$ and C_2 does not contain the letter $(b-1)$. Let $\Gamma(A.a.B.C.D)$ be the word

$$C_1^+.\Pi_{a+1,b-1}.C_2.b.D.$$

For the equivalence, by Lemma 6 (shifting), one has

$$\begin{aligned} a.\Pi_{a+1,b}.C_1 &\equiv \Pi_{a,b}.C_1 \\ &\equiv C_1^+.\Pi_{a,b}. \end{aligned}$$

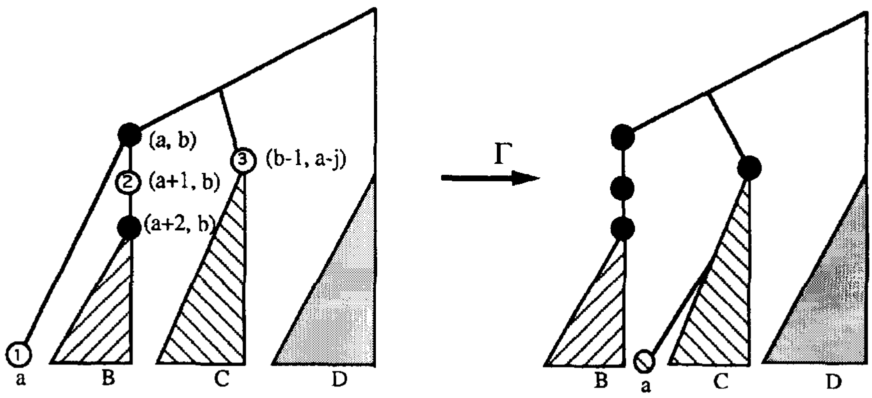

Fig. 7.

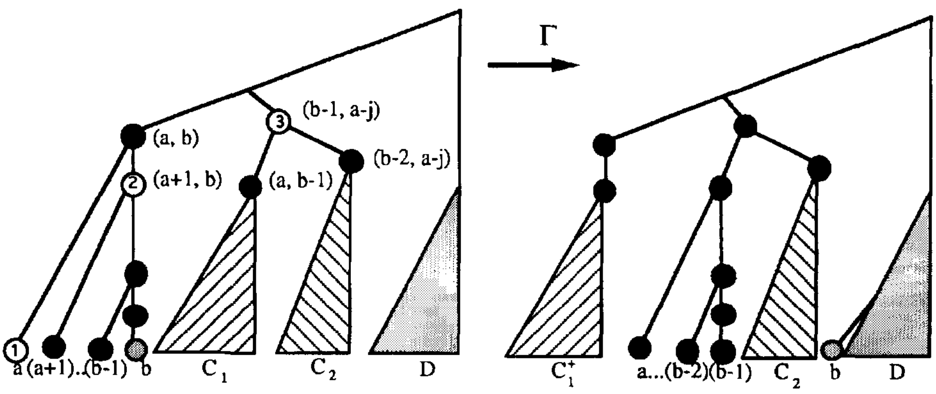

Fig. 8.

As the word C_2 does not contain the letter $(b-1)$, it commutes with the letter b. For the order, the tree of C_1^+ is smaller than the tree of $a.\Pi_{a+1,b}$. Moreover, as the word D is in $\mathscr{W}_{a-j+1,b+k}$ with $k \geq 0$, the tree of D accepts the letter b.
The two other cases are symmetric (Fig. 8). $\square$

We have proved that any reducible word A cannot be normal since one can construct the associated word $\Gamma(A)$. As the sequence of words $A, \Gamma(A), \Gamma^2(A), \ldots,$ is by construction decreasing and as $\ll$ is a wellordering, there exists a finite integer k such that $\Gamma^k(A)$ is irreducible. Let us denote by $\Gamma^*(A)$ this last iterated word. We put $\Gamma^*(A) = A$ when A is irreducible. In the sequel, we will see that $\Gamma^*(A) = \|A\|$ always holds and that every irreducible word is normal. So, the iteration of Γ on any word gives a computation of its normal form.

For the sequel we need to construct "large" irreducible words. Let us show that any irreducible word can be completed on the left in a way that preserves irreducibility.

Lemma 9 (Irreducible completion). *If a word $a.A$ is irreducible then the words $a^2.A$, $(a+1).a^2.A$, $(a-1).a^2.A$ are irreducible too.*

Proof. The word $a^2.A$ has type 1. So, it cannot be reducible. The decomposition of the word $(a+1).a^2.A$ has the form $((a+1), a^2.B, C, D)$. The word $a^2.B$ contains the letter a. Moreover, the word $a^2.B$ cannot be a $\Pi_{a,b}$ since it contains two letters a.

The argument is the same for the word $(a-1).a^2.A$. □

We have constructed several tools on words according to the order $\ll$ that allow us to define the link between special relations and this order $\ll$.

Definition (*Complete words*). Assume that $\lhd$ is a special relation on Σ_n^*. A word B of Σ_n^* is $\lhd$-complete if the relation $A \ll B$ implies $A \lhd B$ for every word A.

The key result of this paper will be that irreducibility, completeness and normality are equivalent notions. We begin with

Proposition 10 (Complete is irreducible). *Assume that $\lhd$ is a special relation on Σ_n^*. Then every $\lhd$-complete word is necessarily irreducible.*

Proof. For a reducible word A, the word $\Gamma(A)$ is before A and is equivalent to A. Then the word $\Gamma(A).1$ is the immediate successor of $\Gamma(A)$ for $\ll$, and it cannot be equal to A (different lengths). We thus have

$$\Gamma(A).1 \ll A.$$

By the property (♣), we have $\varepsilon \lhd 1$. The compatibility with left translations implies $A \lhd A.1$, i.e., $A \lhd \Gamma(A).1$ since A and $\Gamma(A)$ are equivalent. Hence, $\Gamma(A).1 \ll A$ and $A \lhd \Gamma(A).1$ hold. By antisymmetry of $\lhd$, the word A cannot be $\lhd$-complete. □

So, reducible words cannot be $\lhd$-complete. For the converse, we prove by induction on the wellordering $\ll$ that any irreducible word B is $\lhd$-complete since the relation $A \lhd B$ holds for every word A before B. We need several definitions and lemmas. We first consider the case when A and B begin with the same letter.

Definition (*Twins*). Let $a.A$ be a word. A *twin* of $a.A$ is any word before $a.A$ that begins with letter a as well.

So, any twin of a word $a.A$ has the form $a.B$. The compatibility with left translations implies $B \ll A$.

Lemma 11 (Twins). *Let $\lhd$ be a special relation on Σ_n^*. Assume that the word $a.A$ is an irreducible word and that every irreducible word before $a.A$ is $\lhd$-complete. Then $a.B \lhd a.A$ holds for any twin $a.B$ of $a.A$.*

Proof. The word A is irreducible and before $a.A$. By hypothesis, the word A is $\lhd$-complete. As $a.B$ is a twin of $a.A$, $B \ll A$ holds. That implies $B \lhd A$. The compatibility with left translations implies $a.B \lhd a.A$. □

We consider type 1 irreducible words and prove the inductive step for their completeness.

Lemma 12 (Type 1). *Let $\lhd$ be a special relation on Σ_n^*. Assume that the word $a.A$ is an irreducible word of type* 1 *and that every irreducible word before $a.A$ is $\lhd$-complete. Then the word $a.A$ is also $\lhd$-complete.*

Proof. Let $a.A$ be an irreducible word of type 1. Let A' be a word before the word $a.A$. If A' is a twin of $a.A$ then Lemma 11 (twins) implies $A' \lhd a.A$. Assume that A' does not begin with a.

If the word $a.A$ has the form 1^j, then A' is necessarily of the form 1^i with $i < j$. That implies $A' \lhd a.A$.

If the word $a.A$ has the form $a^2.B$, the word A' is necessarily before the word $a.B$ since $a.B \ll A' \ll a^2.B$ implies that A' is a twin of $a.A$. The word $a.B$ is $\lhd$-complete by hypothesis. Thus $A' \lhd a.B$ and $B \lhd a.B$ hold. The compatibility with left translations implies $a.B \lhd a^2.B$. By transitivity, $A' \lhd a^2.B$ holds.

If the word $a.A$ has the form $\Pi_{a,b}.B$ with $a < b$ and B in the submonoid $\{b-1, \ldots, 1\}^*$ then the word A' is necessarily of the form $A_1.A_2$ with A_1 in $\{a+1, \ldots, b\}^*$ and A_2 in $\{b-1, \ldots, 1\}^*$. The property (♣) implies $A' \lhd \Pi_{a,b}$. The compatibility with prefix order implies $\Pi_{a,b} \lhd \Pi_{a,b}.B$. By transitivity, $A' \lhd a.A$ holds. □

In order to prove the inductive step for type 2 irreducible words, we need another notion. We will show by induction on the order $\ll$ that for any irreducible word B of type 2, the relation $A_k \lhd B$ holds for special words A_k that we call the "k-neighbours" of B.

Definition (*Neighbour*). Let $a.A$ be a word of type 1. Let $x(i+2)1^{j+1}$ be the leftmost leaf of A. For every integer k, a word B is *k-neighbour* of $a.A$ if the leftmost leaf of B begins with $x(i+1)k$ (see Fig. 9).

By definition of the order $\ll$ we have immediately

Lemma 13 (Neighbour before). *Every k-neighbour of $a.A$ is before $a.A$ and before every $(k+1)$-neighbour of $a.A$ as well.*

With this notion of neighbour, we can characterize all words that lie before a word of type 2.

Lemma 14 (Predecessors). *Let $a.A$ be a type* 2 *word of Σ_n^*. Every word A' before $a.A$ that is not a twin of $a.A$ lies before any k-neighbour of $a.A$ for k large enough.*

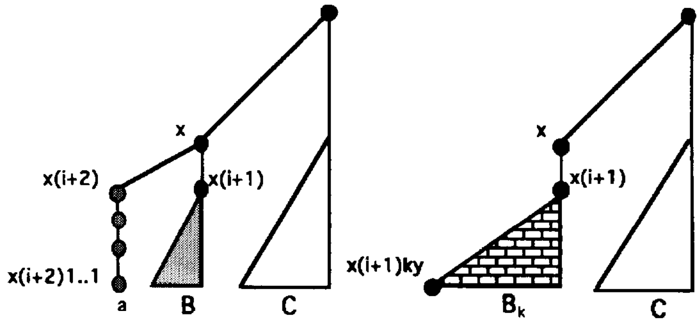

Fig. 9.

Proof. The leftmost leaf of the word $a.A$ has the form $x(i+2)1^{j+1}$. Let w be the leftmost leaf of A'. Then the address w is necessarily at most equal to $x(i+2)1^{j+1}$. Three cases are possible:

(i) For $w \sqsubset x(i+1)$, A' is before every k-neighbour of $a.A$;

(ii) For w of the form $x(i+1)my$, A' is before every k-neighbour of $a.A$ for $k > m$;

(iii) For $w = x(i+2)1^{j+1}$, the word A' begins with a and is a twin of $a.A$. So, we have the result. $\square$

Lemma 15 (Nice neighbour). *Assume that $a.A$ is an irreducible word of type 2 and that every irreducible word before $a.A$ is $\triangleleft$-complete. Then, for every integer k, there exists a word A_k that is a $2k$-neighbour of $a.A$, is irreducible and verifies $A_k \triangleleft a.A$.*

Proof. Assume that the decomposition of the word $a.A$ has the form (a, B, C, D) with B in $\mathscr{W}_{a+1,b}$ with $b > a$ and C in $\mathscr{W}_{b-1,a-j}$ with $j \geq 0$.

Let us put

$$B_0 = (b-1)^2 \cdots (a+2)^2.(a+1)^2.(a+2)^2 \cdots (b-1)^2.b^2,$$

$$C_1 = a^2.(a+1)^2 \cdots (b-2)^2.(b-1)^2,$$

$$C_2 = (b-2)^2.(b-3)^2 \cdots (a-j+1)^2.(a-j)^2.$$

Let us show that $B_0^k.C_1.C_2.D$ serves as the required A_k.

The word $B_0^k.C_1.C_2.D$ is a $2k$-neighbour of $a.A$ since B_0^k contains $2k$ alternations of letters b and $(a+1)$. By definition, the word A ends with $(a-j).D$. By completion, the word $C_1.C_2.D$ is irreducible since $C_1.C_2$ is a sequence of patterns c^2, $(c+1).c^2$ and $(c-1).c^2$. The word $b.C_1.C_2.D$ is irreducible as well since the word $C_1.C_2$ contains the letter $(b-1)$ and is different from $\Pi_{b-1,a-j}$. The word $B_0^k.C_1.C_2.D$ is irreducible. As $a.A$ is irreducible, the word B necessarily contains the letter $(a+1)$ and one of the two following cases occurs:

(i) The word $\Pi_{a+1,b}$ is a strict subword of B.

(ii) The word $\Pi_{a+1,b}$ is B and the word C is not in $\mathscr{W}_{a,b-1}\mathscr{W}_{b-2,a-j}$.

In both cases, for every word C_0 in $\mathscr{W}_{a,b-1}$, $\Pi_{a+1,b}.C_0.C_1.C_2.D \ll B.C.D$ holds. Indeed, for case (i), $\Pi_{a+1,b}.C^*.D \ll B.C.D$ holds for every word C^* in $\mathscr{W}_{b-1,a-j}$, and $C_0.C_1.C_2$ is such a word. For case (ii), $C_0.C_1.C_2.D \ll C.D$ holds. And compatibility with left translations implies $B.C_0.C_1.C_2.D \ll B.C.D$.

The relations $\Pi_{a+1,b}.C_0.C_1.C_2.D \ll B.C.D = A$ and our hypothesis that A is $\lhd$-complete since A is irreducible and before $a.A$, imply $\Pi_{a+1,b}.C_0.C_1.C_2.D \lhd A$. Then compatibility with left translations gives $a.\Pi_{a+1,b}.C_0.C_1.C_2.D \lhd a.A$. But $a.\Pi_{a+1,b}$ is $\Pi_{a,b}$ and $a.\Pi_{a+1,b}.C_0 \equiv C_0^+.a.\Pi_{a+1,b}$ holds. So, applying the previous relation with $C_0 = (B_0^k)^-$ gives $B_0^k.a.\Pi_{a+1,b}.C_1.C_2.D \lhd a.A$. We have to eliminate the subword $a.\Pi_{a+1,b}$ to obtain the result. The word C_1 has the form $a.C_{11}$ with $C_{11} = a.(a+1)^2\ldots(b-2)^2.(b-1)^2$. Since C_1 is not in $\mathscr{W}_{a+1,b-1}$, the word $\Pi_{a+1,b}.C_1.C_2.D$ is irreducible. This word is before $a.A$. So, it is $\lhd$-complete by hypothesis and $C_{11}.C_2.D \lhd \Pi_{a+1,b}.C_1.C_2.D$ holds. The compatibility with left translations implies $B_0^k.a.C_{11}.C_2.D \lhd B_0^k.a.\Pi_{a+1,b}.C_1.C_2.D$, i.e., $B_0^k.C_1.C_2.D \lhd B_0^k.a.\Pi_{a+1,b}.C_1.C_2.D$. Transitivity implies $B_0^k.C_1.C_2.D \lhd a.A$.

So, the word $B_0^k.C_1.C_2.D$ is an irreducible $2k$-neighbour of $a.A$ that verifies $B_0^k.C_1.C_2.D \lhd a.A$. The other decomposition case is analogous. □

We can now prove the inductive step for type 2 irreducible words.

Lemma 16 (Type 2). *Let $\lhd$ be a special relation on Σ_n^*. Assume that the word $a.A$ is an irreducible word of type 2 and that every irreducible word before $a.A$ is $\lhd$-complete. Then the word $a.A$ is also $\lhd$-complete.*

Proof. Assume that A' is before $a.A$. If A' is a twin of $a.A$, then Lemma 11 (twins) implies $A' \lhd a.A$. If A' is not a twin of $a.A$, then by Lemma 14 (predecessors), A' is before every $2k$-neighbour of $a.A$ for k large enough. By Lemma 15 (nice neighbour), there exists a $2k$-neighbour A_k of $a.A$ which is irreducible and which verifies $A_k \lhd a.A$. As this word A_k is before $a.A$, it is $\lhd$-complete by hypothesis. As A' is before this word A_k, we have $A' \lhd A_k$ and by transitivity, we obtain $A' \lhd a.A$. □

The conjunction of Lemma 12 (type 1) and Lemma 16 (type 2) gives by induction the

Proposition 17 (Irreducible is complete). *Assume that $\lhd$ is a special relation on Σ_n^*. Then every irreducible word is necessarily $\lhd$-complete.*

The equivalence between irreducibility and completeness will be the main tool for the following theorem.

Theorem 18 (Coincidence). *Assume that $\lhd$ is a special relation on Σ_n^*. Then*

(i) *$\lhd$ coincides with $\ll$ on normal words;*

(ii) *$\lhd$ is the unique special relation on Σ_n^*;*

(iii) *$\|A\| \ll \|B\|$ implies $\|C.A\| \ll \|C.B\|$;*

(iv) *Every equivalence class contains exactly one irreducible word, namely its* $\ll$-*minimal element. For every word* A *of* Σ_n^*, $\Gamma^*(A) = \|A\|$ *holds;*

(v) $\lhd$ *induces a wellordering of type* $\omega^{\omega^{n-2}}$ *on* B_n^+;

(vi) $\lhd$ *extends the subword ordering.*

Proof. (i) Assume $\|A\| \ll \|B\|$. Since $\|B\|$ is irreducible, by Proposition 17, B is $\lhd$-complete. That implies $\|A\| \lhd \|B\|$, i.e., $A \lhd B$ by compatibility of $\lhd$ with braid relations. Let us show conversely that $A \lhd B$ implies $\|A\| \ll \|B\|$. Assume $\|A\| \not\ll \|B\|$, then $\|B\| \ll \|A\|$ implies $B \lhd A$, and $\|A\| = \|B\|$ implies $A \equiv B$. In both cases, antisymmetry of $\lhd$ excludes $A \lhd B$.

(ii) This is obvious from (i).

(iii) By (i), $\|A\| \ll \|B\|$ implies $A \lhd B$. The compatibility with left translations implies $C.A \lhd C.B$. By (i), we obtain $\|C.A\| \ll \|C.B\|$.

(iv) There cannot exist two distinct equivalent irreducible words A and B. Actually, if we have $A \ll B$ (resp. $B \ll A$) then, by (i), we have $A \lhd B$ (resp. $B \lhd A$) and the antisymmetry of $\lhd$ excludes $A \equiv B$. As every normal word is irreducible, $\Gamma^*(A) = \|A\|$ holds for every A.

(v) From (i), the type of $\lhd$ is the type of $\ll$ restricted to the set of normal words of Σ_n^*. Let us call *perfect* a word in which the degree of every address (except leaves) is at least 2. It is obvious that perfect words are irreducible and therefore normal and that the restriction of the order $\ll$ to perfect words has maximal type $\omega^{\omega^{n-2}}$, since one can consider perfect trees that are as big as we want under any address.

(vi) Since $\lhd$ is compatible with left translations, it is sufficient to verify that $\lhd$ extends the suffix order. From (i), it is sufficient to verify $\|A\| \ll \|a.A\|$ for every word A and every letter a. The proof uses induction on the wellordering $\ll$. For A empty, this is clear from (♣). Consider a nonempty word A. Assume that, for every word A' before A and for every letter a, $\|A'\| \ll \|a.A'\|$ holds. We show the relation $\|A\| \ll \|a.A\|$ for every letter a. Assume that the word $a.\|A\|$ is irreducible. By point (iv), we have $\Gamma^*(a.\|A\|) = \|a.A\| = a.\|A\|$ and we obtain $\|A\| \ll \|a.A\|$. Assume that the word $a.\|A\|$ is reducible. We study every possible case.

Case 1: The decomposition of the word $a.\|A\|$ is (a, B, C, D) and B does not contain the letters $(a+1)$ and $(a-1)$.

As the word $C.D$ is before $\|A\|$, it is before A as well. By hypothesis, we have $\|C.D\| \ll \|a.C.D\|$. By point (iii), $\|B.C.D\| \ll \|B.a.C.D\|$ holds. As $a.B \equiv B.a$ implies $\|B.a.C.D\| = \|a.B.C.D\|$, we have

$$\|B.C.D\| \ll \|a.B.C.D\|.$$

Case 2: The decomposition of the word $a.\|A\|$ is $(a, \Pi_{a+1,b}, C_1.C_2, D)$ with C_1 in $\mathscr{W}_{a,b-1}$ and C_2 in $\mathscr{W}_{b-2,a-j}$. As the word A is normal, necessarily the word C_1 is not in $\mathscr{W}_{a+1,b-1}$. It thus contains a letter a and it is therefore of the form $C_{11}.a.C_{12}$ with C_{11} not containing the letter a (i.e., belonging to $\{a+1,\ldots,b-2,b-1\}^*$) and C_{12} in $\mathscr{W}_{a,b-1}$. As the word $C_{12}.C_2.D$ is before $C.D$, by compatibility with left translations, the word $\Pi_{a+2,b}.C_{12}.C_2.D$ is before $\Pi_{a+2,b}.C.D$ and hence before $\|A\|$ and before A.

By hypothesis, we have

$$\|\Pi_{a+2,b}.C_{12}.C_2.D\| \ll \|\Pi_{a+1,b}.C_{12}.C_2.D\|.$$

From (iii), we have

$$\|C_{11}^{+}.(a+1).a.\Pi_{a+2,b}.C_{12}.C_2.D\| \ll \|C_{11}^{+}.(a+1).a.\Pi_{a+1,b}.C_{12}.C_2.D\|.$$

As C_{11} is in $\{a+1,\dots,b-2,b-1\}^*$, by Lemma 6 (shifting) we have

$$\begin{aligned} C_{11}^{+}.(a+1).a.\Pi_{a+2,b}.C_{12}.C_2.D &\equiv C_{11}^{+}.(a+1).\Pi_{a+2,b}.a.C_{12}.C_2.D \\ &\equiv C_{11}^{+}.\Pi_{a+1,b}.a.C_{12}.C_2.D \\ &\equiv \Pi_{a+1,b}.C_{11}.a.C_{12}.C_2.D \\ &\equiv B.C.D, \end{aligned}$$

$$\begin{aligned} C_{11}^{+}.(a+1).a.\Pi_{a+1,b}.C_{12}.C_2.D &= C_{11}^{+}.(a+1).\Pi_{a,b}.C_{12}.C_2.D \\ &\equiv C_{11}^{+}.\Pi_{a,b}.a.C_{12}.C_2.D \\ &\equiv \Pi_{a,b}.C_{11}.a.C_{12}.C_2.D \\ &\equiv a.B.C.D. \end{aligned}$$

Finally, we have

$$\|B.C.D\| \ll \|a.B.C.D\|.$$

The other decomposition case is analogous. □

Our combinatorial characterization of the wellordering on positive braids proves its uniqueness, but not its existence, which remains dependent on Dehornoy's construction. However, the theorem of coincidence (18) asserts that the only possible special relation on Σ_n^* is the restriction of our ordering $\ll$ to normal words. So, a direct proof of the latter relation being a special relation would be sufficient to establish the existence result. It happens that several points in the definition of a special relation are satisfied by the restriction of $\ll$ to normal words. Actually, the only presently open question is the compatibility with left translations, i.e., the fact that $\|A\| \ll \|B\|$ implies $\|C.A\| \ll \|C.B\|$. We conjecture that a direct combinatorial proof of this latter implication exists. Observe that such a proof would make the present construction nicely self-contained, and that it would also provide a new proof for the antireflexivity property for left distributive systems.

On the other hand, the notion of a normal form gives a (new) algorithm for the word problem on positive braids. To compare two positive braids, one can compute their respective images under Γ^*, which we proved are unique. We conjecture that this algorithm is quadratic with respect to the length of the braids words. Presently, we have completed a proof only for braids of three strands. In this particular case however, one even obtains a linear time bound for the computation of the normal form.

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